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Brief communication

Approximate solutions of non-Newtonian flows over a swarm of bubbles

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1. Introduction

Motion of bubbles in non-Newtonian fluids is important because it occurs in many industrially important situations, such as in chemical and biochemical engineering. Numerous investigations concerning various aspects of the bubble motion have been reported in the literature and have been reviewed in Chhabra (1993). The free surface cell model (Happel, 1958) has been proven to be an excellent model in analyzing slow flow of non-Newtonian fluids in multiparticle systems (Mohan and Raghurman, 1976a,b; Kawase and Ulbrecht, 1981; Chhabra and Raman, 1984). The cell model has been extended to the flow of both Newtonian and non-Newtonian fluids in multiple droplets or bubbles system (Gar-Or and Waslo, 1968; Bhavaraju et al., 1978; Kawase and Ulbrecht, 1981; Jarzebski and Malinowski, 1986, 1987a,b; Gummalam and Chhabra, 1987; Gummalam et al., 1988; Zhu and Deng, 1994; Zhu, 1995; Zhu, 2001). Jarzebski and Malinowski used the variational principles to obtain the upper and lower bounds on the drag coefficient for a power law fluid (Jarzebski and Malinowski, 1986) and for a Carreau fluid (Jarzebski and Malinowski, 1987a). The variational principles were also used to calculate the rising velocity of spherical bubbles in a power law fluid (Gummalam and Chhabra, 1987) and in a Carreau fluid (Gummalam et al., 1988). The numerical simulations have also been used to study the flow of droplets and bubbles in power law fluids (Zhu and Deng, 1994) and Carreau fluids (Zhu, 1995; Zhu, 2001) by using the cell model. Chhabra (1998) used the cell model for estimating the rising velocity of a swarm of spherical bubbles in power law liquids at high Reynolds number. Kawase and Ulbrecht (1981)

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developed an analytical solution of power law fluids through multiple particles and multiple bubbles by approximating the second invariant of the rate of deformation tensor using Newtonian results (Hirose and Moo-Young, 1969). Jarzebski and Malinowski (1987b) established the approximate expressions for the drag coefficient and Sherwood number for slow motion of a swarm of Newtonian drops in a power law fluid using a similar approach to that of Kawase and Ulbrecht (1981).

To summarize, previous studies used approximate analytical, the variational bound-form and numerical solutions to solve the problem. The closed form analytical solutions provide deeper insights as to the significance of governing factors. They can be the common tools of many engineering applications, such as creation of simplified engineering models for predictions and sensitivity analyses, separation of the complex process involved in many engineering applications into fewer conceptually more tractable processes, etc. In this study, a simpler and more effective approximate analytical solution for the power law fluids through a swarm of bubbles is developed. The analytical solution for the drag coefficients and rising velocity is derived by assuming that the all flow related quantities for the power law fluid is close to its Newtonian counterparts. The approach used is similar to the technique used in previous investigations (e.g., Hirose and Moo-Young, 1969; Kawase and Ulbrecht, 1981; Jarzebski and Malinowski, 1987b), but further simplifies the approximation procedure. The previous studies typically used Newtonian results to approximate only the components of the rate of deformation tensor. The same problem is also solved using the finite difference method. The developed approximate solutions are compared against the numerical solutions.

2. Problem statement

The rheological behavior of the fluids is represented by a power law model,

$$\tau_{ij} = 2K(2\Pi)^{(n-1)/2} D_{ij} \tag{1}$$

where τ_{ij} is the stress tensor, K is the consistency index, n is the flow behavior index, D_{ij} is the rate of deformation tensor, and the second invariant of the rate of deformation tensor, Π, is given by

$$\Pi = D_{rr}^2 + D_{\theta\theta}^2 + D_{\phi\phi}^2 + 2D_{r\theta}^2 \tag{2}$$

The following dimensionless variables are introduced,

$$\begin{aligned} D_{ij}^* &= \frac{D_{ij}}{(V_0/R)}, \quad \Pi^* = \frac{\Pi}{(V_0/R)^2}, \quad \tau_{ij} = \frac{\tau_{ij}}{K(V_0/R)^n} \\ p^* &= \frac{p}{K(V_0/R)^n}, \quad v_i^* = \frac{v_i}{V_0}, \quad \xi = \frac{r}{R}, \quad \psi^* = \frac{\psi}{V_0 R^2} \end{aligned} \tag{3}$$

where V_0 is the superficial velocity, R is the radius of the bubble, p is the pressure, v_i is the velocity component, r is radial distance, and ψ is the stream function.

The governing equations can be written in the spherical coordinate system as,

$$E^{*^2}\psi^* = \omega^* \xi \sin\theta \tag{4}$$

$$(2\Pi^*)^{(n-1)/2}E^{*^2}(\omega^*\xi\sin\theta)+(n-1)(2\Pi^*)^{(n-3)/2}\left[\frac{\partial\Pi^*}{\partial\xi}\frac{\partial}{\partial\xi}(\omega^*\xi\sin\theta)+\frac{\partial\Pi^*}{\partial\theta}\frac{1}{\xi^2}\frac{\partial}{\partial\theta}(\omega^*\xi\sin\theta)\right]$$
$$=2(1-n)F(\xi,\theta)\sin\theta \tag{5}$$

where,

$$E^{*^2}=\frac{\partial^2}{\partial\xi^2}+\frac{\sin\theta}{\xi^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right) \tag{6}$$

$$F(\xi,\theta)=\frac{\partial}{\partial\xi}\left[\xi D^*_{\xi\theta}(2\Pi^*)^{(n-3)/2}\frac{\partial\Pi^*}{\partial\xi}+D^*_{\theta\theta}(2\Pi^*)^{(n-3)/2}\frac{\partial\Pi^*}{\partial\theta}\right]$$
$$-\frac{\partial}{\partial\theta}\left[D^*_{\xi\xi}(2\Pi^*)^{(n-3)/2}\frac{\partial\Pi^*}{\partial\xi}+\frac{D^*_{\xi\theta}}{\xi}(2\Pi^*)^{(n-3)/2}\frac{\partial\Pi^*}{\partial\theta}\right] \tag{7}$$

where θ is the latitude angle of the spherical coordinate system. The boundary conditions are specified as follows:

On the bubble surface,

$$\xi=1,\quad \psi^*=0,\quad \omega^*=2v^*_\theta \tag{8}$$

and on the outer sphere surface,

$$\xi=s,\quad \psi^*=-\frac{1}{2}\xi^2\sin^2\theta,\quad \omega^*=2(v^*_\theta+\sin\theta)/s \tag{9}$$

where s is the dimensionless radius of the cell related to the gas holdup Φ (i.e., the gas volumetric fraction in the system) by

$$s=\frac{R_1}{R}=\Phi^{-1/3} \tag{10}$$

where R_1 is the radius of the hypothetical fluid envelope.

3. Approximate solutions

When non-Newtonian flow behavior is not very strong (i.e., $|n-1|$ is small), we assume that all flow related variables that are raised to the power of $(n-1)/2$ or multiplied by $(n-1)$ can be evaluated by using the Newtonian solutions as approximation. The procedure used in this study thus uses a wider range of approximation than that used typically in earlier investigations (e.g., Hirose and Moo-Young, 1969; Kawase and Ulbrecht, 1981; Jarzebski and Malinowski, 1987b) in which only the components of the rate of deformation tensor were evaluated using the Newtonian results. Therefore, the assumption in this study is broader and the results are simpler. Substituting the Newtonian results for these terms in Eqs. (4) and (5), one can obtain,

$$E^{*^4}\psi^* = \frac{6n(n-1)}{1-1/s}\xi^{-2}\sin^2\theta \quad (11)$$

The general solution to the above equation can be established as,

$$\psi^* = \left[a_1\xi^4 + a_2\xi^2 + a_3\xi + a_4\xi^{-1} + \frac{n(n-1)}{1-1/s}\xi\ln\xi\right]\sin^2\theta \quad (12)$$

The constants in Eq. (12) should be determined to satisfy the boundary conditions (8) and (9), which leads to,

$$a_1 = \frac{n(n-1)s(1-s^2)}{6(1-s^5)(s-1)} \quad (13a)$$

$$a_4 = \frac{n(n-1)s}{6(s-1)} - a_1 s^5 \quad (13b)$$

$$a_2 = \frac{-s/2 - n(n-1)s\ln s/(s-1) - a_1(s^3-1) - a_4(s^{-2}-1)}{s-1} \quad (13c)$$

$$a_3 = -a_1 - a_2 - a_4 \quad (13d)$$

The dimensionless surface pressure is calculated using the following relationship,

$$p_s^*(\theta) = p_1^* + \int_0^\theta (2\Pi^*)^{(n-1)/2}\left[\omega^* + \frac{\partial\omega^*}{\partial z} + \frac{(n-1)D_{\xi\xi}^*}{\Pi^*}\frac{\partial\Pi^*}{\partial\theta}\right]\Bigg|_{z=0} \mathrm{d}\theta \quad (14)$$

$$\text{where } z = \ln\xi \text{ and } p_1^* = 2\int_0^{\ln s}(2\Pi^*)^{(n-1)/2}\left[\frac{\partial\omega^*}{\partial\theta} - \frac{(n-1)D_{\xi\xi}^*}{2\Pi^*}\frac{\partial\Pi^*}{\partial z}\right]\Bigg|_{\theta=0}\mathrm{d}z \quad (15)$$

The flow drag on the bubble is then given by

$$F_\mathrm{D} = 2\pi R^2\left[\int_0^\pi(-p+\tau_{rr})_{r=R}\cos\theta\sin\theta\,\mathrm{d}\theta - \int_0^\pi(\tau_{r\theta})_{r=R}\sin^2\theta\,\mathrm{d}\theta\right] = 2\pi R^2 K(V_0/R)^n D_0 \quad (16)$$

$$\text{where} \quad D_0 = \int_0^\pi[-p_s^*(\theta) + (\tau_{rr}^*)_{z=0}]\cos\theta\sin\theta\,\mathrm{d}\theta \quad (17)$$

The correction factor of drag coefficient for non-Newtonian behavior is,

$$Y_\mathrm{D} = \frac{C_\mathrm{D}}{16/R_e'} = 2^{n-2}D_0 \quad (18)$$

where C_D is the drag coefficient, R_e' is Reynolds number for power law fluids, defined as,

$$R_e' = \frac{\rho V_0^{2-n}(2R)^n}{K} \tag{19}$$

By using the approximate solutions, the correction factor of drag coefficient is then,

$$Y_D = \frac{2^{n-1}}{n(n+2)}\left\{\frac{3}{(1-s^{-1})^2}\right\}^{(n-1)/2}\left\{8(1-n)a_1+2(1+2n)a_3-12(1-n)a_4+\frac{(n-1)(1-n-4n^2)}{1-s^{-1}}\right\} \tag{20}$$

The governing Eqs. (4) and (5) are also solved using the finite difference technique. The implementation of the numerical technique is similar to Zhu (2001) where the numerical scheme was developed for a similar problem for the Carreau model.

4. Results and discussion

In order to show that the differences of the key flow related variables (such as vorticity, rate of deformation tensor) between non-Newtonian and Newtonian fluids are indeed small and therefore the assumption used can be justified, the second invariant of the rate of deformation tensor on the bubble surface for both the Newtonian and the power law fluids is shown in Fig. 1(a) and the surface vorticity is shown in Fig. 1(b). Results show that only in the case of very small gas

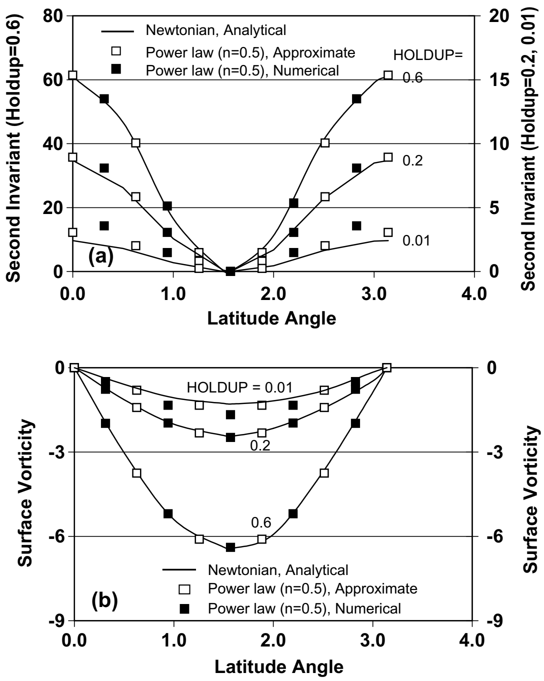

Fig. 1. Comparison of flow variables on the bubble surface using various approaches, (a) surface second invariant of rate of deformation tensor, and (b) surface vorticity.

holdup, is the discrepancy of the second invariant between the Newtonian and the power law fluids noticeable. In the medium range of the gas holdup, Newtonian flow field is indeed a very good approximation for that of power law fluids.

4.1. Drag coefficient

The Comparison of the drag coefficient (in terms of the correction factor, Y_D), from the approximate solutions with that from the full numerical results is shown in Fig. 2. As expected, the drag coefficient Y_D decreases as the flow behavior index, n, decreases due to stronger shear-thinning (decreasing viscosity) behavior of the power law fluids. Results show that the drag coefficient increases as the gas holdup increases and the degree of this augmentation becomes less significant as the shearthinning effect becomes stronger. Therefore it may be concluded that the shearthinning behavior of fluids reduces the holdup effect on the drag coefficient. It can be observed that the analytical solutions compare quite favorably with the full numerical results, especially for the medium gas holdup range.

4.2. Rising velocity

The expression for the ratio of the bubble swarm velocity to single-bubble velocity in power law fluids can be obtained as,

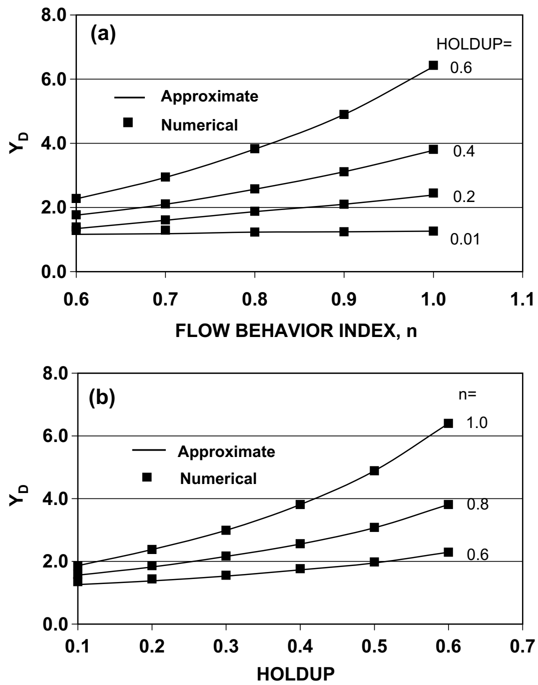

Fig. 2. Comparison of approximate solutions against full numerical results, (a) drag coefficient vs. Newtonian flow behavior index, n; and (b) drag coefficient vs. gas holdup, Φ.

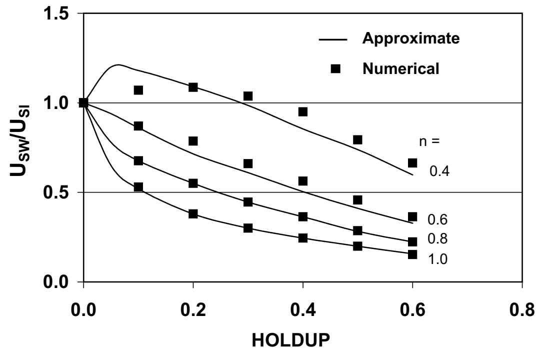

Fig. 3. Comparison of the ratio of swarm velocity to the single-bubble velocity between the approximate solutions and the full numerical results.

$$\frac{U_{\mathrm{SW}}}{U_{\mathrm{SI}}} = \left[\frac{Y_{\mathrm{DSI}}}{Y_{\mathrm{D}}}\right]^{1/n} \tag{21}$$

where Y_{DSI} denotes the drag coefficient for a single bubble rising in a power law fluid.

It can then be obtained from the approximate solution that

$$\frac{U_{\mathrm{SW}}}{U_{\mathrm{SI}}} = (1-s^{-1})\left\{\frac{n(19-2n-8n^2)}{3(1-s^{-1})[8(1-n)a_1+2(1+2n)a_3-12(1-n)a_4]+3(n-1)(1-n-4n^2)}\right\}^{1/n} \tag{22}$$

The comparison of the ratio of the bubble swarm velocity to the single-bubble velocity between the approximate solutions and the full numerical results is shown in Fig. 3. Results show that up to about $n = 0.6$ they agree very well. At $n = 0.4$, the rising velocity increases initially and reaches a maximum value at about $\Phi = 0.2$ due to the nullifying of two opposite mechanisms, i.e., the hindrance effect of the bubbles and the shearthinning effect of the fluids. The approximate solutions slightly overestimate the velocity ratio at small gas holdup and underestimate it at high gas holdup. The analytical prediction for the rising velocity is slightly less accurate than that for the drag coefficient, since any inaccuracy of the approximate solutions is magnified by a power of $1/n$ in predicting the rising velocity.

To the best of our knowledge, no sufficient details regarding a swarm of bubbles in non-Newtonian fluids were given in experimental studies reported in the literature to enable a direct comparison between the present study and experimental results. However, the physical resemblance of a swarm of bubbles to an assemblage of spherical rigid particles, previously successfully solved using the cell model gives grounds for anticipating a fair predictive capability for the solutions developed in this study.

5. Concluding remarks

The drag coefficient and rising velocity of a swarm of bubbles through power law fluids under the slow flow conditions are obtained analytically by using the widely used free surface cell model.

Results show that the simple approximate analytical solutions in the present study predict the drag coefficient and the rising velocity of the bubbles with good accuracy, for n as small as 0.5. The non-Newtonian disturbance to the key flow fields, such as the flow velocity and the vorticity, is small. The small flow field disturbance permits the approximation of flow related quantities using Newtonian counterparts, which forms the basis for this study. While the analytical solutions developed in this study generally agree very well with the full numerical results, the approximate solutions slightly overestimate the rising velocity and underestimate the drag coefficient at small gas holdup.

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